International Journal of Thermophysics, Vol. 14, No. 4, 1993

A Note on the Turbulent Energy as a Parameter of Turbulent Flow[1]

L. N. Persen[2]

Received December 2, 1992

The art of "modeling turbulence" is a needed tool in the construction of computer codes for turbulent flows. The state to which this art has been developed is inadequate, and quotations from authoritative sources support this point of view. The energy contained in the turbulent fluctuations, i.e., the turbulent energy, is often used as a parameter in the modeling process. The present article attempts to examine this quantity as it is being created, transported, and dissipated. For this purpose experimental evidence from the author's own experiments (free jets), as well as theoretical conclusions from the elementary deductions of the basic equations, the concept of turbulent potential flow, and a general solution to the Navier–Stokes–Reynolds equations, is drawn to attention. Recirculating flow is given special attention. The paper concludes with recommendations for principles that must be satisfied if improved modeling is to be achieved. These principles are necessary; whether they are also sufficient is open to question.

KEY WORDS: turbulence modeling; recoverable work; potential turbulent flow; round jet; line source for turbulent energy; apparent dissipation; dissipation; redistribution of energy.

1. INTRODUCTION

It is a fact that the turbulent energy, i.e., the mean kinetic energy ($q = \frac{1}{2}\rho\overline{u'u'} + \frac{1}{2}\rho\overline{v'v'} + \frac{1}{2}\rho\overline{w'w'}$) of the turbulent motion (u', v', and w'), is used as a parametric quantity in most numerical schemes used to analyze turbulent fluid motion.[3] It is, furthermore, realized that the modeling used for the basic equation for this quantity is inadequate in many of its present

[1] Paper dedicated to Professor Joseph Kestin.

[2] Department of Mechanical Engineering, University of Trondheim, N-7034 Trondheim, NTH, Norway.

[3] Definitions of symbols are given under Nomenclature.

0195-928X/93/0700-0637$07.00/0

forms. Reference is made here to the review article by McCrosky *et al.* [1], from which the following quotations are cited:

> Except for the limitations of turbulence modeling, the showcase problems of 1974 can be solved routinely today.
>
> The validity (of numerical simulations) is essentially determined by the turbulence modeling....
>
> The problem of turbulence modeling is probably the one with the least optimism,....

Attention is also drawn to the test cases presented by Bouffinier and Grandotta [2] at the International Symposium on Refined Flow Modeling and Turbulence Measurements: 17 contributors had computed the specified flow using the k-ε model. The results were confronted with experimental evidence and velocity profiles as well as distributions of turbulent kinetic energy were used as criterion for accuracy. The following is quoted from the summary:

> Concerning the mean velocity quite all the results compared well together and with the experiment. Several calculations gave similar results for the turbulent quantities, but all differed clearly from the measurements.

It seems clear that the state of the art presented above prompts a closer scrutiny of the procedures behind turbulent modeling. The purpose of this presentation is to draw attention to physical realities which ought to be considered in the modeling procedure.

2. SOME FACTS ABOUT ENERGY

The energy balance of a system is governed by laws of nature. The first law of thermodynamics is such a law. Since the system's kinetic energy is one form of energy to be incorporated into the energy balance the system's velocity must be found. Thus a second law of nature enters the picture: Newton's laws of motion.

If the system under consideration is chosen as a fluid element with sides δx, δy, and δz, one may write the equations of motion as

$$\begin{aligned}
\rho \frac{Du}{Dt} &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho X \\
\rho \frac{Dv}{Dt} &= \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho Y \qquad (1) \\
\rho \frac{Dw}{Dt} &= \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho Z
\end{aligned}$$

where the notations are those used by Kestin [3]. Attention is drawn here to the fact that the stresses acting on the element are unspecified. Neither the Stokes' relations nor any other phenomenological relation has been introduced, and consequently the deductions to be made with basis in these equations may be considered quite general. However, the body forces acting on the element will be assumed deducible from a potential (V), i.e.,

$$\vec{F}=\vec{i}X+\vec{j}Y+\vec{k}Z=-\text{grad}\,(V) \tag{2}$$

If equations in Eq. (1) are multiplied consecutively by the velocity components (u, v, w) and added, one obtains the mechanical energy equation

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\vec{v}^2\right)=\rho\vec{F}\cdot\vec{v}+W_1 \tag{3}$$

where W_1 is the work done by the stresses as the system moves without being deformed:

$$W_1=\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)u+\left(\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right)v$$
$$+\left(\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_z}{\partial z}\right)w \tag{4}$$

By means of the equation of continuity (expressing the conservation of mass principle),

$$\frac{D\rho}{Dt}+\rho\,\text{div}(\vec{v})=0 \tag{5}$$

the mechanical energy equation may be reformulated as

$$\underbrace{\frac{D}{Dt}\left(\frac{1}{2}\rho\vec{v}^2\right)+\frac{1}{2}\rho\vec{v}^2\,\text{div}(\vec{v})}_{\text{Change of kinetic energy}}\qquad\underbrace{+\rho\vec{v}\cdot\text{grad}(V)}_{\text{Change of potential energy}}\qquad=W_1 \tag{6}$$

This equation has been deduced in detail, even though it is part of most introductory texts, for the purpose of emphasizing the following points.

1. Equation (6) shows that the stresses acting on the element under consideration may perform work that may be positive as well as negative. Whether it is one or the other depends on the elements

change in potential and kinetic energy. This means, in other words, that *the stresses, which may be entirely viscous, may perform recoverable work.*

2. The mechanical energy equation expresses an energy balance that exists due to (or as a consequence of) the equations of motion. *The energy that is being dissipated (in the mechanical sense) and converted to heat (considered as a non-mechanical form of energy) does not enter this equation.*
3. These two statements regarding the mechanical energy equation and the dissipated energy are quite generally valid and will be called upon when subsequently discussing the energy of the turbulent motion.

The total work (W) done by the stresses acting on the system (the element) may be expressed as

$$W = W_1 + W_2 = W_1 + \sigma_x \frac{\partial u}{\partial x} + \tau_{yx} \frac{\partial v}{\partial x} + \tau_{zx} \frac{\partial w}{\partial x} + \tau_{xy} \frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y}$$

$$+ \tau_{zy} \frac{\partial w}{\partial y} + \tau_{xz} \frac{\partial u}{\partial z} + \tau_{yz} \frac{\partial v}{\partial z} + \sigma_z \frac{\partial w}{\partial z} \tag{7}$$

This equation defines the work (W_2) done by the stresses as the system (the fluid element) deforms without moving. This work consists of the work done as the system undergoes a shape-true change of volume and the work done as the system undergoes a volume-true change of shape. This latter part represents the dissipation in a Newtonian fluid. The first part is zero in the case of an incompressible fluid.

Finally, the first law of thermodynamics is applied to the system. It states that the system's change of energy is equal to the heat introduced into it minus the work done by the system on its surroundings. The system's internal energy is $\hat{u}$ and the first law of thermodynamics may be formulated as

$$\underbrace{\frac{D}{Dt}\left(\rho\hat{u} + \frac{1}{2}\rho\vec{v}^2\right) + \left(\rho\hat{u} + \frac{1}{2}\rho\vec{v}^2\right)\operatorname{div}(\vec{v}) + \rho\vec{v}\cdot\operatorname{grad}(V)}_{\text{Change in internal, kinetic, and potential energy}} = -\operatorname{div}(\vec{q}) + W \tag{8}$$

where the heat flux into the system is given as

$$\vec{q} = \vec{i}q_x + \vec{j}q_y + \vec{k}q_z \tag{9}$$

This equation may be reduced in complexity by observing the energy balance expressed through the mechanical energy equation, Eq. (6), and the continuity equation, Eq. (5). The simplified form of Eq. (9) will then be

$$\rho \frac{D\hat{u}}{Dt} = -\operatorname{div}(\vec{q}) + W_2 \tag{10}$$

This equation is generally valid and attention is again drawn to the fact that no phenomenological relation has been introduced. The only limitation is in the assumed conservation of mass which excludes nuclear reactions and in the adopted heat flux which excludes internal heat sources.

The energy equation (the first law of thermodynamics) in its form as Eq. (10) shows that the work done (W_2) as the system undergoes a volumetrue change of shape is creating a rise in the systems internal energy, i.e., dissipation is always creating heat (Newtonian fluid). The work done as the system moves without deforming does not enter the energy equation, Eq. (10).

The final conclusion is that an equation which expresses the system's energy balance and contains the dissipation must also contain terms which express the change in the system's internal energy.

The facts about energy highlighted here are well-known to the profession. Yet a number of modeling efforts when the turbulent energy is concerned do not seem to take account of these facts, as demonstrated subsequently.

3. RECOVERABLE WORK

The introduction of the concept of viscous forces performing recoverable work may be illustrated by a simple example.

Assume an infitely extended flat plate above which an infinitely extended viscous (Newtonian) fluid is located. Assume, furthermore, that the flat plate performs a rectilinear harmonic oscillation. The velocity of the plate is thus given by U_{plate}:

$$U_{\text{plate}} = U_0 \cos(\omega t) \tag{11}$$

The velocity field set up in the fluid is such that it may be characterized by all streamlines being parallel to the x-axis. Thus,

$$u(y, t) \neq 0, \qquad v = 0, \qquad w = 0 \tag{12}$$

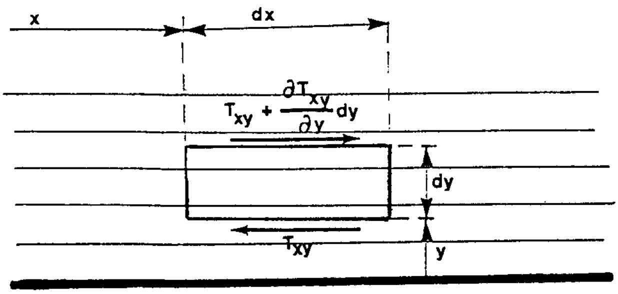

Fig. 1: The element near an oscillating plate.

Figure 1 shows a fluid element in the neighborhood of the plate. The only forces performing work during a given time interval δt are the shear stresses τ_{xy}. The total work done (per unit time) will be

$$W\,dx\,dy = -\tau_{xy} u\,dx + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}\,dy\right)\left(u + \frac{\partial u}{\partial y}\,dy\right)dx \tag{13}$$

from which one obtains

$$W = \tau_{xy}\frac{\partial u}{\partial y} + u\frac{\partial \tau_{xy}}{\partial y} \tag{14}$$

The two types of work done are here easily identified:

$$\begin{aligned} W_1 &= u\frac{\partial \tau_{xy}}{\partial y} = \mu u \frac{\partial^2 u}{\partial y^2} \\ W_2 &= \tau_{xy}\frac{\partial u}{\partial y} = \mu\left(\frac{\partial u}{\partial y}\right)^2 \end{aligned} \tag{15}$$

where the Stokes hypothesis for the relation between the stresses and the rate of strain (velocity field) has been introduced.

The solution to the flow field is given by Kestin [3] and is recalled here:

$$u(y, t) = U_0 e^{-\eta}\cos(\omega t - \eta) \tag{16}$$

where

$$\eta = y\sqrt{\frac{\omega}{2\nu}} \tag{17}$$

Introduction of Eq. (16) into Eq. (15) will give

$$W_1 = -\tfrac{1}{2}\rho U_0^2 \omega e^{-2\eta} \sin(2(\omega t - \eta)) \tag{18}$$

The mechanical energy equation, Eq. (3) or Eq. (6), will, in the present case, take the form

$$\frac{D}{Dt}\left(\frac{1}{2}\rho \vec{v}^2\right) = W_1 \tag{19}$$

It is thus seen that the kinetic energy of the system (the fluid element) will oscillate, that the work done by the stresses is stored in the kinetic energy as an increase when the work is positive, and that this work is recovered as the kinetic energy decreases

This is a point to be recalled when discussing the modeling of the turbulent energy.

4. EXPERIMENTAL EVIDENCE I

The energy of the turbulent motion, i.e., the kinetic energy contained in the turbulent fluctuations (in the Reynolds sense), has been measured in numerous flow situations. It is not the intention of this presentation to give an extensive survey of these but rather to extract from the literature a few results which throw light on the problem at hand.

Before looking at the details of experimental evidence it is necessary to ascertain that the uncertainty in experimental determination of turbulent fluctuations is clearly understood. This uncertainty originates from the frequency limitations of the equipment used, and the measured fluctuations may be as much a function of the characteristics of the equipment as they may reflect physical reality. Thus, only when the average values may be shown not to be dependent on the frequency limitations of the equipment may physical significance be attached to the measured value.

The case of free turbulence as given in the free jet into still fluid is a case which is well suited for the study of the turbulent energy. Persen [4] reports measurements of the quantities $\sqrt{\overline{u'u'}}$ and $\sqrt{\overline{v'v'}}$ in the case of a plane (two-dimensional jet). Guided by the theoretical approach, which incorporates a general solution, he was able to show that both quantities decay in the same way with downstream distance and that, consequently, a similarity condition characterizes the turbulent flow field. Thus, even in the near field of the jet, similarity (within an acceptable accuracy) exists both for the velocity field and for the turbulent energy. This has a profound significance as will be demonstrated subsequently.

The experiment referred to in Ref. 4 incorporates the use of two jet opening geometries: one sharp-edged and one rounded opening. This leads to a difference in the characteristics of the jet flow as sought illustrated in Fig. 2. This difference does not influence the similarity properties of the velocity profiles but does, indeed, influence the actual values of the normalizing distances such as the jet's half-width and the characteristic length downstream. This is a significant observation for the present investigation because it indicates that the absolute values of the different terms of the turbulent energy are directly dependent on their boundary conditions, i.e., on conditions at the point of origin of the fluctuations.

Figure 3 shows the measured distribution of the quantity $\sqrt{\overline{u'u'}}$, where the crossflow coordinate η is defined as

$$\eta = y/b_{1/2}(x) \tag{20}$$

and where $b_{1/2}$ is the jet's half-width determined from the measured velocity profiles. The way in which the decay occurs with downstream distance is suggested described by the expression

$$\left.\begin{matrix}\sqrt{\overline{u'u'}}\\ \sqrt{\overline{v'v'}}\end{matrix}\right\} = B(x)\, e^{-\eta^2/2} + A(x)(e^{-0.9(\eta-1)^2} + e^{-0.9(\eta+1)^2}) \tag{21}$$

where x is the downstream distance and ξ is its dimensionless form:

$$\xi = (x - x_0)/L_{\text{char}} \tag{22}$$

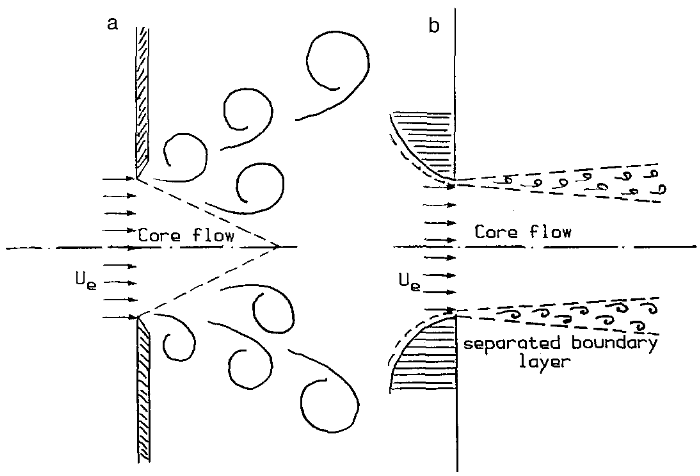

Fig. 2. Sketch illustrating the difference in the flow field with the sharp-edge (a) and the rounded-edge (b) nozzle.

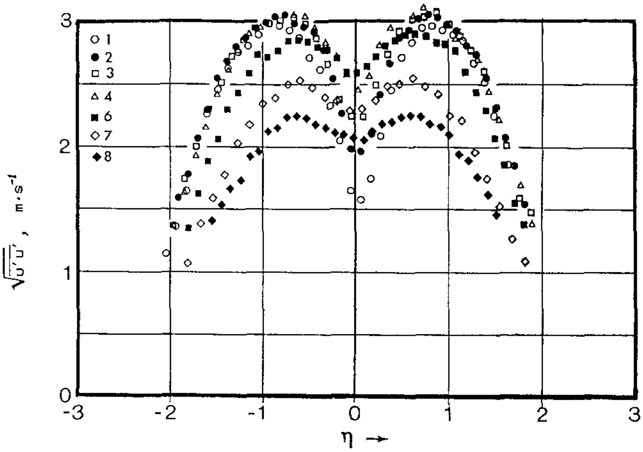

Fig. 3. Profiles showing the crossflow distribution of $\sqrt{\overline{u'u'}}$ at different downstream locations (plane jet).

A proposed hypothesis about the propagation of a disturbance is used to suggest the following expression for the unknown function $A(x)$:

$$A(x) = (1 + \xi^2)^{-1/4} \tag{23}$$

Since the same behavior is found for both the $\sqrt{\overline{u'u'}}$ and the $\sqrt{\overline{v'v'}}$ data, one may by inference conclude that a corresponding behavior must apply for the turbulent energy. Thus a study of Eq. (21) will give information also on the latter quantity.

5. THE TURBULENT POTENTIAL FLOW

Attention is drawn here to the fact that potential flow is characterized by a condition of kinematic nature being placed on the flow field, i.e., the vector field representing the flow be irrotational. This concept may of course also be applied to the turbulent flow as a condition on the mean flow vector field. The conditions for this to be feasible were investigated by Persen [5]. It should be emphasized that whether the flow is laminar or turbulent, potential flow does not imply that energy is not being dissipated. It does, however, mean that the recoverable work done by the mean viscous forces as the fluid element moves is zero. If the reason for this is examined, one finds that three conditions lead to this conclusion:

1. The viscous stresses are linked to the flow field through the Stokes hypothesis.

2. The mean velocity vector's divergence is zero (continuity of an incompressible fluid).
3. The velocity vector is assumed to be derivable from a flow potential (irrotationality).

Comparable conditions are not valid for the Reynolds stresses. This indicates, therefore, the possibility that the recoverable work done by the Reynolds stresses may be different from zero even when the mean flow field is irrotational.

In Ref. 5 a necessary (but not necessarily sufficient) condition for the existence of a turbulent potential flow field is given. Assuming this condition to be met, one may proceed to the introduction of the stress functions χ_1, χ_2, and χ_3, from which the resulting force on the element may be deduced. This in turn leads to the introduction of the "stress potential" Λ, defined as

$$\Lambda = \tfrac{1}{2}\rho\vec{q}^2 + \tfrac{1}{2}\nabla^2(\chi_1 + \chi_2 + \chi_3) \tag{24}$$

The energy equation for the flow (stationary case) may then be formulated:

$$\underbrace{\frac{D}{Dt}\left(\frac{1}{2}\rho\vec{v}^2\right)}_{\text{I}} + \underbrace{\rho\vec{v}\cdot\text{grad}(V)}_{\text{II}} = \underbrace{-\vec{v}\cdot\text{grad}(p)}_{\text{III}} + \underbrace{\vec{v}\cdot\text{grad}(\Lambda)}_{\text{IV}} \tag{25}$$

where V is the potential for the body forces acting on the element. Each term in this equation is to be interpreted as follows:

(i) change per unit volume and unit time of the element's kinetic energy,

(ii) change per unit volume and unit time of the element's potential energy,

(iii) work done (per unit volume and unit time) by the pressure acting on the element as it moves without deforming, and

(iv) work done (per unit volume and unit time) by the Reynolds stresses as the element moves without deforming. (The corresponding work done by the viscous forces is zero).

The details of the further deductions are not recapitulated here but the following results are drawn to attention.

a. If the flow under consideration is a recirculating one, i.e., the streamlines form closed loops, the value of Λ may change along a streamline, but any positive change must be compensated by an equal negative change so that the total change is equal to zero when returning to the point of departure. Since Λ depends on the turbulent energy, a corresponding statement applies to this quantity.

b. If the situation of a uniform potential flow with parallel steamlines is considered, the equations of motion in integrated form simplify to

$$p+\Lambda=\text{constant} \tag{26}$$

Since Λ in this case is dependent exclusively on the turbulent energy, this reflects the fact that the pressure may act as a "storage" of turbulent energy and that this storage may be both emptied and filled as the turbulent energy changes. The way in which this happens is unknown.

6. A GENERAL CASE

The Navier–Stokes–Reynolds equations for turbulent flows may be formulated as follows:

$$\begin{aligned}
&\frac{\partial}{\partial x}(\bar{\sigma}_x^0-p-\rho\overline{u'u'}-\rho\bar{u}^2)+\frac{\partial}{\partial y}(\bar{\tau}_{xy}-\rho\overline{u'v'}-\rho\bar{u}\bar{v})\\
&\qquad+\frac{\partial}{\partial z}(\bar{\tau}_{xz}-\rho\overline{u'w'}-\rho\bar{u}\bar{w})=0\\
&\frac{\partial}{\partial x}(\bar{\tau}_{xy}-\rho\overline{u'v'}-\rho\bar{u}\bar{v})+\frac{\partial}{\partial y}(\bar{\sigma}_y^0-p-\rho\overline{v'v'}-\rho\bar{v}^2)\\
&\qquad+\frac{\partial}{\partial z}(\bar{\tau}_{yz}-\rho\overline{v'w'}-\rho\bar{v}\bar{w})=0 \qquad (27)\\
&\frac{\partial}{\partial x}(\bar{\tau}_{xz}-\rho\overline{u'w'}-\rho\bar{u}\bar{w})+\frac{\partial}{\partial y}(\bar{\tau}_{yz}-\rho\overline{v'w'}-\rho\bar{v}\bar{w})\\
&\qquad+\frac{\partial}{\partial z}(\bar{\sigma}_z^0-p-\rho\overline{w'w'}-\rho\bar{w}^2)=0
\end{aligned}$$

where again the notations used are those of Kestin [3]. In this form the equations may be conceived of as d'Alembert's formulation of dynamic equilibrium, and as shown by Persen [4], a general solution to the equations may be found. However, the general solution that can be established does not solve the closure problem. The solution does, on the other hand, give information on the distribution of the turbulent energy in special cases such as the one treated by Persen [4] (the plane turbulent jet). This is referenced in Section 4. Supplementary experimental evidence is examined in Section 7.

At present, attention is drawn to the form of Eqs. (27), which suggests that a relation between the elements of the two matrices **A** and **B** might be found:

$$\mathbf{A}=\begin{bmatrix} \bar{u}\bar{u},\ \bar{u}\bar{v},\ \bar{u}\bar{w} \\ \bar{u}\bar{v},\ \bar{v}\bar{v},\ \bar{v}\bar{w} \\ \bar{u}\bar{w},\ \bar{v}\bar{w},\ \bar{w}\bar{w} \end{bmatrix},\quad \mathbf{B}=\begin{bmatrix} \overline{u'u'},\ \overline{u'v'},\ \overline{u'w'} \\ \overline{u'v'},\ \overline{v'v'},\ \overline{v'w'} \\ \overline{u'w'},\ \overline{v'w'},\ \overline{w'w'} \end{bmatrix} \tag{28}$$

This suggested relationship is also made plausible by the contention that the energy in the main flow and the energy in the turbulent fluctuations decay very much in the same way under conditions where the effect of wall regions are negligible. A further study of this is referenced below.

7. EXPERIMENTAL EVIDENCE II

The free jet, i.e., the flow out of a nozzle into still fluid with no guiding walls, represents a flow which is very well suited for an experimental examination of the distribution of energy in the flow. However, the fact that the finite width of a slit sets a limit to the region in which a plane jet maintains its two-dimensionality makes the plane jet less suited for experimental investigation. Presently, therefore, the round jet as investigated by Persen [6] is examined and the following observations are made:

1. The jet is produced from a cutoff tube of 42-mm diameter.
2. The distance (z_0) downstream of the nozzle beyond which the velocity profiles exhibit self-similarity is determined from the downstream distribution of the jet's half-width.
3. In the region of self-similarity ($z > z_0$) the jet's half-width ($b_{1/2}$) and the centerline velocity (U_c) follow the general expressions

$$b_{1/2} = b_{1/2}^{(0)}(1+\xi^2)^{1/2} \tag{29}$$

$$U_c = U_c^{(0)}(1+\xi^2)^{-n} \tag{30}$$

where ξ is the dimensionless downstream distance and η is the dimensionless radial distance:

$$\xi = (z-z_0)/L_{\text{char}} \tag{31}$$

$$\eta = r/b_{1/2} \tag{32}$$

4. The velocity components in the self-similar region are given as

$$\bar{v}_z = U_c f(\eta) \tag{33}$$

$$\bar{v}_r = -\frac{U_c^{(0)} b_{1/2}^{(0)}}{L_{\text{char}}} 2\xi(1+\xi^2)^{-n-1/2} F(\eta) \tag{34}$$

where $f(\eta)$ is the so far unspecified mathematical expression for the hat curve representing the self-similar velocity profile and where

$$F(\eta) = (-n+1)\frac{1}{\eta}\int_0^\eta sf(s)\,ds - \frac{1}{2}\eta f(\eta) \tag{35}$$

5. The total kinetic energy (E_z) in the jet's axial velocity at any given profile is obtained by integration:

$$E_z = \int_0^\infty \tfrac{1}{2}\rho\bar{v}_z^2 2\pi r\,dr \tag{36}$$

or dimensionless

$$E_z^* = E_z/\pi\rho[U_c^{(0)} b_{1/2}^{(0)}]^2 = (1+\xi^2)^{-2n+1}\int_0^\infty \eta[f(\eta)]^2\,d\eta \tag{37}$$

One observes here that the integral in this expression must be a constant and that, consequently, $n \geqq \frac{1}{2}$ since the energy cannot be increasing with downstream distance. If $n = \frac{1}{2}$ the energy contained in the axial velocity remains constant downstream, and the apparent decay is caused exclusively by a redistribution of the energy. For $n > \frac{1}{2}$ a real dissipation occurs in addition to the redistribution.

The total kinetic energy (E_r) in the jet's radial velocity at any given profile will be

$$E_r = \int_0^\infty \tfrac{1}{2}\rho\bar{v}_r^2 2\pi r\,dr \tag{38}$$

or dimensionless

$$E_r^* = E_r/\pi\rho[U_c^{(0)} b_{1/2}^{(0)}]^2$$

$$= \left(\frac{b_{1/2}^{(0)}}{L_{\text{char}}}\right)^2 4\xi^2(1+\xi^2)^{-2n}\int_0^\infty \eta[F(\eta)]^2\,d\eta \tag{39}$$

Again, conditions must be placed on the asymptotic behavior of the function $f(\eta)$ such that the integral in Eq. (39) is a constant. One may then conclude that the energy contained in the radial velocity initially does not decay in the same way as that contained in the axial velocity. Eventually, though, they both decay as ξ^{-4n+2} for large values of ξ.

These analytic considerations are now supplemented by experimental results. The spread of the jet's half-width as indicated by Eq. (29) is experimentally supported by data as shown in Fig. 4. The data for the centerline velocity are exhibited in Fig. 5. By a best-fit procedure the following results are found:

$$\begin{aligned} L_{\text{char}} &= 224.5 \text{ mm} \\ z_0 &= 117 \text{ mm} \\ b_{1/2}^{(0)} &= 22.8 \text{ mm} \\ U_c^{(0)} &= 9.43 \text{ m} \cdot \text{s}^{-1} \\ n &= 0.5441 \end{aligned} \tag{40}$$

It is important to notice that $n > \frac{1}{2}$ indicates that the apparent decay of the energy in the jet's main flow is caused by both a redistribution downstream and a real dissipation.

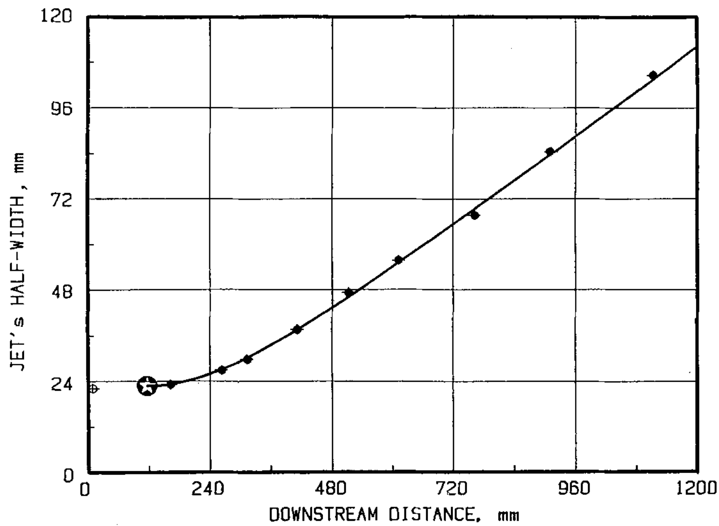

Fig. 4. The spread of the plane jet with downstream distance illustrated by the variation in the jet's half-width.

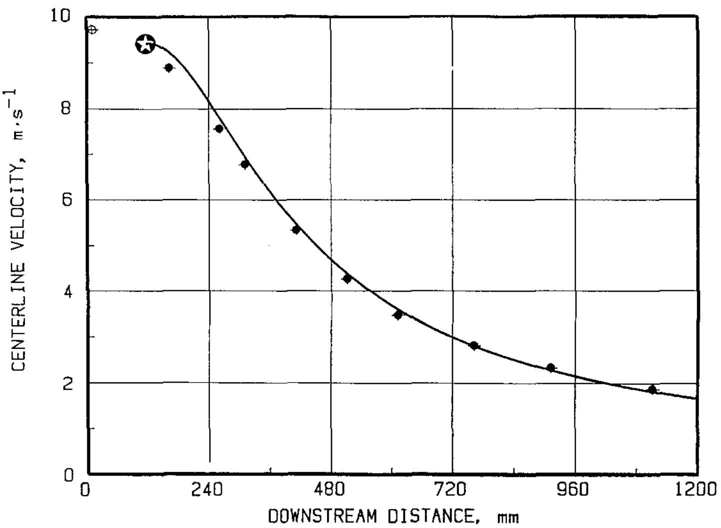

Fig. 5. The decay in the centerline velocity with increasing downstream distance.

Having in this way described the energy distribution in the main flow of the jet, it becomes important to investigate the distribution of the energy in the turbulent fluctuations.

Foremost in these considerations must be the question of whether or not the dimensionless distances ξ and η, determined with origin in the velocity data, also may be used in the description of the turbulent energy distribution. The answer lies in Fig. 6, where a series of profiles of the measured turbulent energy ($\sqrt{\overline{u'u'}}$) in the axial fluctuation component is plotted using the dimensionless crosswise variable η. There is an original skewness in the profile which is damped out downstream.

The main features of the profiles (when plotted in this way) are illustrated by the plotting of the two maximum amplitudes and their location as done in Figs. 7 and 8. It is seen how the two maxima gradually become equal with increasing downstream distance, whereas their location becomes constant.

To describe the behavior of the profiles the following expression is adopted for the profile:

$$\sqrt{\overline{u'u'}} = A(\xi)\, H(\eta) \tag{41}$$

where

$$H(\eta) = e^{-\lambda(\eta - \eta_0)} + e^{-\lambda(\eta + \eta_0)} \tag{42}$$

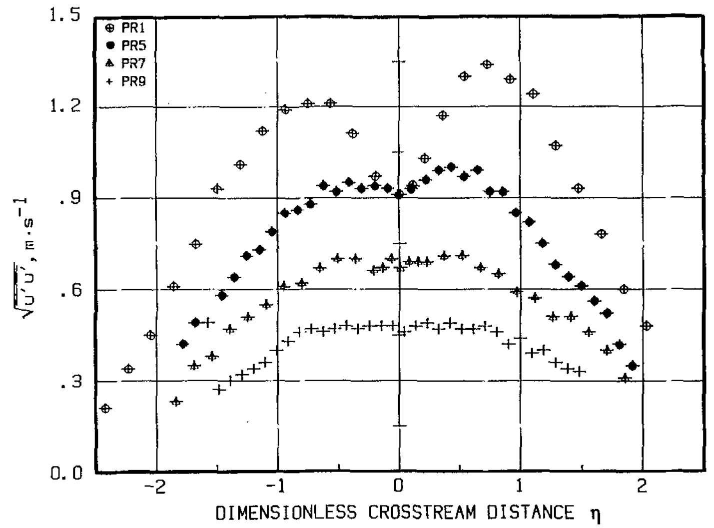

Fig. 6. Crosswise distribution of $\sqrt{\overline{u'u'}}$ in the round jet.

These analytic expressions give symmetrical profiles and Fig. 9 shows how the data of profile 5 are approximated by the analytic expressions with the parameters λ and η_0 having been determined by a best-fit procedure.

When such procedures have been completed for all profiles, one will have data showing how $A(\xi)$ varies with downstream distance. This means

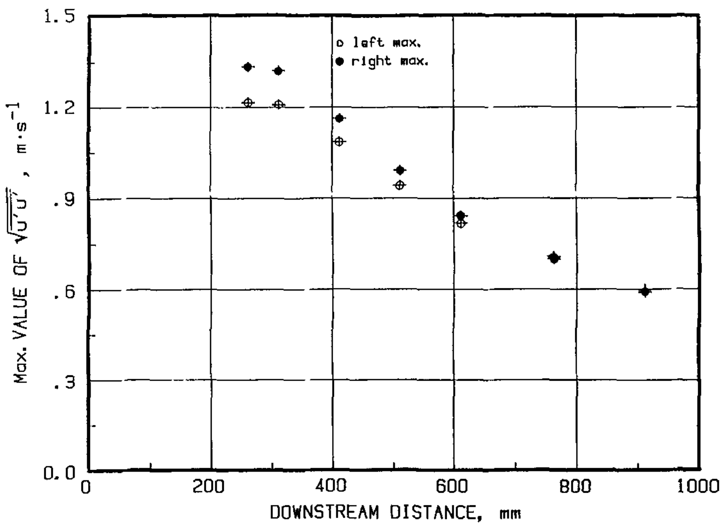

Fig. 7. Downstream decay of the two maxima in Fig. 6.

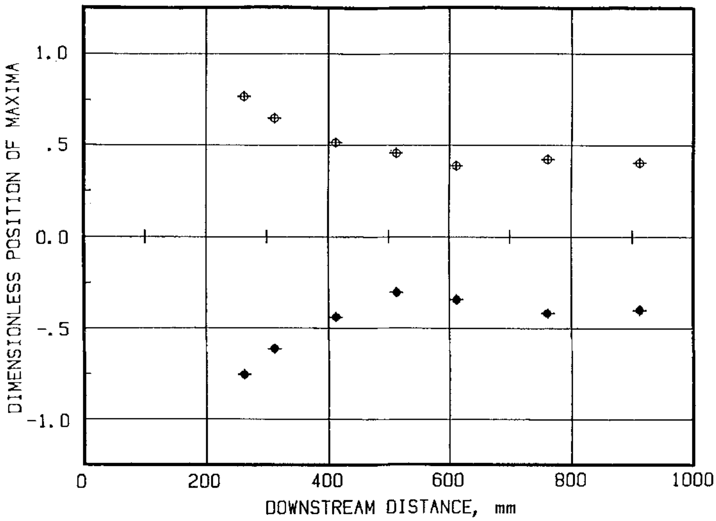

Fig. 8. Downstream variation in the position of the two maxima in Fig. 6.

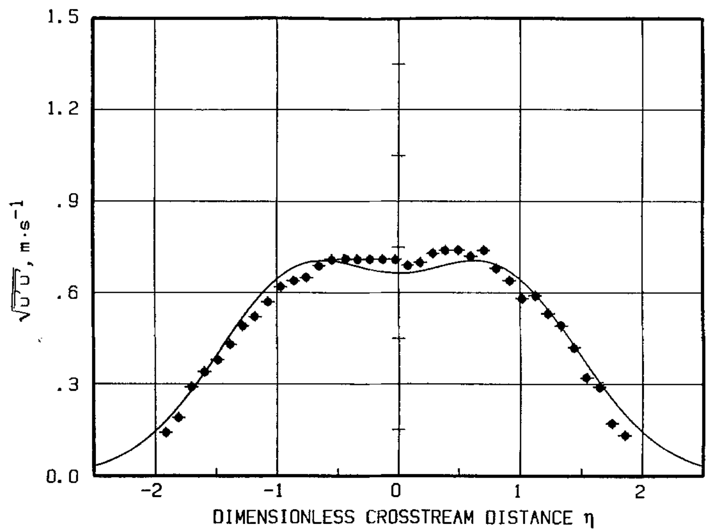

Fig. 9. Analytic expression for the profiles in Fig. 6 compared with experimental results.

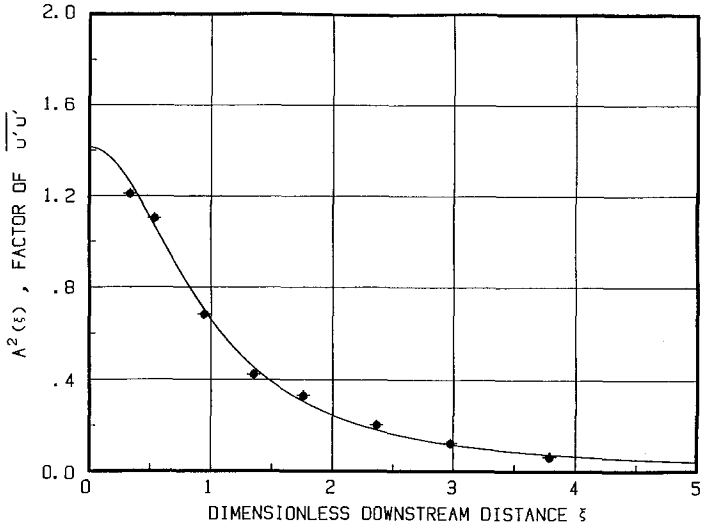

Fig. 10. Decay with downstream distance (dimensionless) of the maximum value in the turbulent energy.

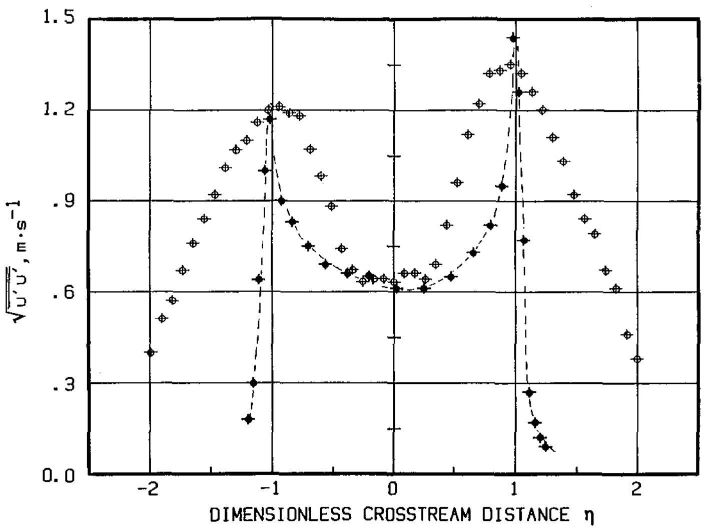

Fig. 11. Profiles close to the nozzle illustrating the origin of the two maxima in the fluctuation profiles as coming from a singularity at $\eta = 1$.

that the apparent decay in the energy in the turbulent fluctuations in the axial direction may be founc by assuming $A(\xi)$ to follow the law

$$A^2(\xi) = A_0^2(1 + \xi^2)^m \tag{43}$$

where m is determined by a best-fit procedure. The result is given in Fig. 10 and m is determined to

$$m = -1.0921 \qquad (2n = 1.0882) \tag{44}$$

This is very close to the value $(2n)$ determined for the energy in the main flow and supports the contention that the turbulent energy behaves very much like the energy in the main flow.

Before proceeding to draw conclusions a certain feature of the flow must be drawn to attention. The maxima of the fluctuation profiles indicate a disturbance originating at $\eta = 1$, i.e., at the wall of the tube representing the nozzle. For that purpose a profile was measured as close to the nozzle as practically possible (10 mm). This profile and the corresponding profile measured in the region before the profile start to become self-similar $(z < z_0)$ as shown in Fig. 11. It is observed that the profiles are those to be expected with a singularity occurring at the wall of the tube exit. This is where the flow field may be conceived of as having a nonanalytic boundary condition. The observation serves as an illustration to the creation of intense turbulent activity at a wall. Further comments are advanced in the concluding section.

8. CONCLUSION

Before actually commenting on turbulence modeling, it may be worthwhile to give some thought to whether or not turbulent modeling really is all that important. Lumley [7] organized a workshop in 1989 with the specific objective of discussing where turbulence research is headed. It is undeniable that many claims have been made over the years that the final answer is now near. Coherent structures, chaos, etc., are milestones along this road. Have the Navier–Stokes–Reynolds equations really lost their significance? Is a solution to these equations really of no interest any more? And should turbulence modeling be left at its present state because it serves the profession well as it is?

The answer to these questions will have to be ambiguous because it depends on the basic point of view of the one who answers. Many computer codes in use today are based on turbulence modeling that renders good service in design in aerospace applications. By carefully fine-tuning the constants that the modeling provides, one will, for defined purposes, get very satisfactory results. So, why bother? However, if the aim is to understand turbulence as a physical phenomenon, the answer is that turbulence modeling needs improvement.

Assuming that adequate solutions to the Navier–Stokes–Reynolds equations could be found: Would, then, all questions in connection with turbulent flow be answerable? The answer to this question is no. The probability of a certain fluctuation being likely to exceed a certain limit is a feature of the flow that is beyond the grasp of these equations. Even so, it is the opinion of the present author that our understanding of turbulent flow would be greatly enhanced if turbulence modeling could be brought to a state of refinement so that the closure gap could be bridged. Great consolation is found in the fact that Lumley [7] seems to support this idea.

One must contemplate the feasibility of finding a solution to bridge the closure gap. A generally valid relationship between the Reynolds stresses and the flow field in any form that would be applicable both in the free field and in the boundary layer regions seems to be out of the question. A reasonable approach would therefore seem to be to establish experimental support for turbulence modeling that remains valid in as great a generality as can reasonably be expected.

The preceding sections have presented some scattered pieces of information which are thought to give ideas to be applied in the search for improved turbulence modeling. Most of the modeling done today is based on the rather elementary idea that the diffusion process can be conceived of as a "flux = turbulent transport coefficient × gradient to a driving field" (Fourier analogy). In most cases the turbulent energy appears as the parameter. Persen [8] discussed the shortcommings of this approach, and the purpose of the present considerations is to suggest an alternative approach where needed.

Sections 2 and 3 place emphasis on the mechanical energy equation and draw attention to the fact that mean viscous stresses can do "recoverable" work and that this is especially important in recirculating flows. In potential flow this work is zero (Section 5).

Section 4 references experimental results which indicate that turbulent energy is transported very much like kinetic energy in the mean flow, a result that is further supported by the theoretical evidence referred to in Section 6.

Section 7 references further experimental evidence in strong support of the fact that turbulent energy is being conveyed downstream in a free flow field in the same way as the kinetic energy in the main flow.

This leads to the following postulates for an improved modeling:

a. Distinction must be made between cases with near wall flows and free flow fields.

b. When the free flow field is approximately a potential flow, the model must reflect that the mean viscous stresses do not perform recoverable work. Whether the Reynolds stresses do or do not is uncertain and is left for experimental investigations.

c. If the flow is stationary and recirculating (streamlines are closed loops), the total work is dissipated. The dissipated energy must be supplied from outside the system. This makes the k-ε model inadequate in its elementary form.

d. The turbulent boundary layer is characterized by its two regions: the wall region and the wake region. In the latter the development downstream resembles that of the free turbulent jet. Consequently the suggested relation in Eq. (28) ought to apply and the model should reflect this. It is emphasized that the relation needed to make Eq. (28) into a proper phenomenological relation is not specified here.

e. Whenever the modeling implies deduction of a differential equation for the turbulent energy, the fact that turbulent energy may be created by a "point or line source" (Fig. 11) must be specified in the boundary conditions.

It is clear that these considerations will have to be followed up by specifications, where these have been left open here.

A final remark that may give food for thought is; Why is it that the flow in a turbulent boundary layer on a flat plate can be adequately described analytically without any consideration being given to the behavior of the turbulent energy or its boundary conditions?

NOMENCLATURE

A_0	Constant
$b_{1/2}$	Jet's half-width
$b_{1/2}^{(0)}$	Jet's half-width at $z = z_0$
E_z	Kinetic energy contained in the jet's axial velocity at a given profile
E_r	Kinetic energy contained in the jet's radial velocity at a given profile
$f(\eta)$	Dimensionless velocity profile [$f(0) = 1$]
$F(\eta)$, $H(\eta)$	Defined functions
L_{char}	Jet's characteristic length
m, n	Exponents

p	Pressure
q	Kinetic energy in the turbulent fluctuations
$\vec{q}$	Heat flux
q^2	$= \overline{u'u'} + \overline{v'v'} + \overline{w'w'}$
r, ϕ, z	Cylindrical coordinates
t	Time
$\hat{u}$	Internal energy
u, v, w	Velocity components
$\bar{u}, \bar{v}, \bar{w}$	Mean velocity components
$\bar{v}_r, \bar{v}_\phi, \bar{v}_z$	Mean velocity components
U_0	Constant
U_{plate}	Plate's velocity
$U_c^{(0)}$	Centerline velocity at $z = z_0$
X, Y, Z	Components of body force
W	Total work done by surface stresses
W_1	Recoverable work done by surface stresses
W_2	Dissipated work
z_0	Downstream distance from the nozzle beyond which self-similar velocity profiles occur
ν	Fluid's kinematic viscosity
ρ	Fluid's density
σ_x, σ_y, σ_z	Normal stresses
τ_{xy}, τ_{yz}, τ_{zx}	Shear stresses
σ_x^0, σ_y^0, σ_z^0	Normal stresses with the pressure removed
$\eta = y/\sqrt{\omega/2\nu}$, $\eta = y/b_{1/2}$	Dimensionless Crossflow coordinate
η_0	Constant
χ_1, χ_2, χ_3	Stress functions
Λ	"Stress potential"

REFERENCES

1. W. J. McCroskey, P. Kutler, and J. O. Bridgeman, AGARD Structures and Materials Panel Specialist' Meeting on "Transonic Unsteady Aerodynamics and its Aeroelastic Applications," Sept. 3–5 (1984), Paper No. 9.
2. C. Bouffinier and M. Grandotto, Commissariat à l'Énergie Atomique, Cadarache, France, Proc. Int. Symp. Refined Turbulence Model. Turbulence Measure., C11, University of Iowa, Ames, Sept. 16–18 (1985).
3. J. Kestin, *Boundary Layer Theory* (translation of H. Schlichting's book) (McGraw–Hill, New York, Toronto, London, 1960), pp. 75–76.
4. L. N. Persen, AGARD Conference Proceedings No. 308, Symposium on Fluid Dynamics of Jets with Applications to V/Stol, Lisbon, Portugal, Nov. 2–5 (1981).
5. L. N. Persen, DGLR-Vortrag Nr. 81-013, Jahrestagung der DGLR, Aachen, May 11–14 (1981).
6. L. N. Persen, *Turbulence Research*, Trends in Heat Mass & Momentum Transfer, 2 (1992), pp. 223–235.
7. J. L. Lumley, *in* Proceedings of a Workshop Held at Cornell University, March 22–24, 1989, Lecture Notes in Physics (Springer Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1990), pp. 1–4, 49–56.
8. L. N. Persen, in *Progress in Aerospace Sciences, Vol. 23*, No. 3 (Pergamon Press, Oxford, New York, Beijing, Frankfurt, Sao Paolo, Sydney, Tokyo, Toronto, 1986), pp. 167–183.